

Inhomogeneous Condensates in Planar QED*

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Abstract

We study the formation of vacuum condensates in $2 + 1$ dimensional QED in the presence of inhomogeneous background magnetic fields. For a large class of magnetic fields, the condensate is shown to be proportional to the inhomogeneous magnetic field, in the large flux limit. This may be viewed as a *local* form of the *integrated* degeneracy-flux relation of Aharonov and Casher.

1 Introduction

Parity- and flavor-symmetry breaking aspects of $2 + 1$ dimensional QED have been the subject of much research in recent years. This subject has applications in planar electron systems and also provides a deep analogue of certain features of symmetry breaking in $3 + 1$ dimensional theories relevant for particle physics [1, 2, 3]. An important focus of these studies is the question of induced charges and spins, which are also related to induced vacuum condensates. Gusynin et al [4] have shown recently that a uniform background magnetic field acts as a catalyst for dynamical flavor symmetry breaking in $2 + 1$ dimensions. A key part of this argument is the appearance of a nonzero vacuum flavor condensate, in the limit of zero fermion mass, in the presence of a uniform background magnetic field of strength B :

$$\langle 0 | \bar{\psi} \psi | 0 \rangle_{m \rightarrow 0} = -\text{sign}(m) \frac{B}{2\pi} \quad (1)$$

While much can be learned from this constant B field case, in order to include dynamical gauge fields it is desirable to have a more complete understanding of this phenomenon for

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more general background electromagnetic fields. As (1) refers to a uniform B field, it is of course consistent with the *integrated* relation

$$\int d^2x \langle 0 | \bar{\psi}(\vec{x}) \psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} = -\text{sign}(m) \frac{1}{2\pi} \int d^2x B(\vec{x}) = -\text{sign}(m) \Phi \quad (2)$$

(where Φ is the net magnetic flux), which is essentially Landau's degeneracy-flux relation [5], and which was extended to inhomogeneous magnetic fields by Aharonov and Casher [6, 7]. Much important work has been done exploring the detailed global aspects of this integrated result (2), and relating it to mathematical index theorems [8, 9, 10, 11, 12].

The emphasis of this paper is rather different - here we investigate the extent to which (2) may be viewed as a *local* relation

$$\langle 0 | \bar{\psi}(\vec{x}) \psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} \stackrel{?}{=} -\text{sign}(m) \frac{1}{2\pi} B(\vec{x}) \quad (3)$$

when the background magnetic field is inhomogeneous. This, and the closely related issue of induced spin, have been addressed for the special case of an Aharonov-Bohm flux string magnetic field [13]. In this paper, we consider the formation of a vacuum condensate in the presence of a more general spatially inhomogeneous static background magnetic field. We present some illustrative examples in which the condensate may be evaluated explicitly, and then we show that for a large class of inhomogeneous magnetic fields the condensate is proportional to the magnetic field [just as in (3)], but only in the large flux limit.

In Section II we give a brief review of vacuum condensates in $2 + 1$ dimensional QED. Section III contains two explicit examples of particular inhomogeneous magnetic fields and Section IV contains the general asymptotic analysis for radial magnetic fields. Finally, we conclude with some brief comments.

2 Vacuum Condensates in Planar QED

Consider a parity invariant model of $2 + 1$ dimensional quantum electrodynamics with fermionic Lagrange density

$$\mathcal{L}_F = \bar{\psi} (i\Gamma^\mu D_\mu - m) \psi \quad (4)$$

Here ψ is a four-component spinor and the gamma matrices Γ^μ belong to a 4×4 *reducible* representation

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}, \quad (5)$$

where the 2×2 *irreducible* gamma matrices γ^μ are given by

$$\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned}\gamma^1 &= i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ \gamma^2 &= i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\end{aligned}\tag{6}$$

These gamma matrices are normalized as $\{\Gamma^\mu, \Gamma^\nu\} = -2g^{\mu\nu}\mathbf{1}$, where the flat Minkowski metric is $g^{\mu\nu} = \text{diag}(-1, 1, 1)$. The covariant derivative operator is $D_\mu = \partial_\mu - iA_\mu$, where for notational convenience we have absorbed a factor of “ e ” into the gauge field A_μ . This model is invariant under the generalized parity transformation [14, 2, 3]

$$x^1 \rightarrow -x^1, \quad A_1(x^1, x^2) \rightarrow -A_1(-x^1, x^2), \quad \psi(x^1, x^2) \rightarrow \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \psi(-x^1, x^2)\tag{7}$$

and in the massless limit, $m \rightarrow 0$, has a global $U(2)$ flavor symmetry corresponding to the interchange of the two 2×2 irreducible representations.

We consider static background gauge fields and work in the Weyl ($A_0 = 0$) gauge. Then the Dirac equation, $(i\Gamma^\mu D_\mu - m)\psi = 0$, block diagonalizes as

$$\begin{pmatrix} E - m & -(D_1 - iD_2) & 0 & 0 \\ (D_1 + iD_2) & E + m & 0 & 0 \\ 0 & 0 & E + m & -(D_1 - iD_2) \\ 0 & 0 & (D_1 + iD_2) & E - m \end{pmatrix} \psi = 0\tag{8}$$

which illustrates the fact that this reducible representation model is equivalent to a theory describing two species of two-component spinors, one with mass $+m$ and the other with mass $-m$. Without loss of generality, we choose m to be positive, and we also choose the net magnetic flux to be positive.

The upper 2×2 sub-block of the Dirac equation (8), corresponding to the positive mass species, is solved by a two-component spinor

$$\chi = e^{-iEt} \begin{pmatrix} f \\ -\frac{(D_1 + iD_2)}{E + m} f \end{pmatrix}\tag{9}$$

when $E \neq -m$, and where $f(x, y)$ is a solution of the two-dimensional partial differential equation

$$-(D_1 - iD_2)(D_1 + iD_2)f = \alpha^2 f\tag{10}$$

with $\alpha^2 = E^2 - m^2$. Thus, there are solutions of positive and negative energy, $E = \pm\sqrt{\alpha^2 + m^2}$. When $|E| \neq m$, we can write these solutions (including the appropriate normalization factors) as

$$\psi_{\{1\}}^{(\pm)} = e^{\mp i|E|t} \sqrt{\frac{|E| \pm m}{2|E|}} \begin{pmatrix} f \\ \mp \frac{(D_1 + iD_2)}{|E| \pm m} f \\ 0 \\ 0 \end{pmatrix}$$

$$\psi_{\{2\}}^{(\pm)} = e^{\mp i|E|t} \sqrt{\frac{|E| \mp m}{2|E|}} \begin{pmatrix} 0 \\ 0 \\ f \\ \mp \frac{(D_1 + iD_2)}{|E| \mp m} f \end{pmatrix} \quad (11)$$

where f satisfies the Schrödinger-like equation (10). Here, the subscript $\{1\}$ refers to species $\{1\}$ which corresponds to mass $+m$, while species $\{2\}$ has mass $-m$. The superscripts (\pm) refer to the positive and negative energy solutions.

The threshold states, with $|E| = m$, are special and must be specified separately. Indeed, already from (11) we see that for species $\{1\}$ we can have a positive energy solution with $|E| = m$, but the $1/\sqrt{|E| - m}$ factor excludes a negative energy threshold state of this form. By contrast, for species $\{2\}$ we can have a negative energy solution of this form with $|E| = -m$, while the $1/\sqrt{|E| - m}$ factor now excludes a positive energy threshold state. This imbalance leads to an asymmetry in the spectrum of states, and this asymmetry is the ultimate source of the interesting symmetry breaking effects in planar QED.

The explicit threshold states are

$$\psi_{\{1\}}^{(0+)} = e^{-imt} \begin{pmatrix} f^{(0)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \psi_{\{2\}}^{(0-)} = e^{+imt} \begin{pmatrix} 0 \\ 0 \\ f^{(0)} \\ 0 \end{pmatrix} \quad (12)$$

where $f^{(0)}(x, y)$ satisfies the *first-order* threshold equation¹

$$(D_1 + iD_2)f^{(0)} = 0 \quad (13)$$

Now expand the fermion field operator in terms of creation and annihilation operators for an orthonormal set of positive and negative energy modes from (11) and (12):

$$\Psi = \sum_{i=1}^2 \sum_n \sum_p [b_{n,p} \psi_{(i),n,p}^{(+)} + d_{n,p}^\dagger \psi_{(i),n,p}^{(-)}] \quad (14)$$

¹Note that we have excluded the potential threshold states of the form $\psi_{\{1\}}^{(0-)} = e^{imt} \begin{pmatrix} 0 \\ g^{(0)} \\ 0 \\ 0 \end{pmatrix}$ and $\psi_{\{2\}}^{(0+)} = e^{-imt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ g^{(0)} \end{pmatrix}$, where $g^{(0)}$ satisfies $(D_1 - iD_2)g^{(0)} = 0$, because if the solutions to (13) are normalizable then these $g^{(0)}$ solutions are not, and vice versa [6, 7].

where $b_{n,p}$ and $d_{n,p}$ are fermionic annihilation operators. The label n refers to the eigenvalue α_n^2 of the equation (10) and hence specifies the energy, while the label p distinguishes between degenerate states. Note that both n and p may take discrete and/or continuous values, depending on the equations (10) and (13) respectively. Also, note that the sum over the species in (14) is understood to include the positive energy threshold states for species $\{1\}$ and the negative energy threshold states for species $\{2\}$.

The vacuum expectation value $\langle 0|\bar{\Psi}\Psi|0\rangle \equiv \langle 0|\bar{\Psi}^\dagger\Gamma^0\Psi|0\rangle$ is then given by

$$\langle 0|\bar{\Psi}\Psi|0\rangle = -\sum_p |f_p^{(0)}|^2 - 2m \sum_{n>0} \sum_p \frac{|f_{n,p}|^2}{|E|} \quad (15)$$

where the first term on the RHS only involves threshold states, while the second term involves all states with $|E| > m$. In the massless limit the second term vanishes and the condensate is simply

$$\langle 0|\bar{\Psi}(\vec{x})\Psi(\vec{x})|0\rangle|_{m\rightarrow 0} = -\sum_p |f_p^{(0)}(\vec{x})|^2 \quad (16)$$

The minus sign on the RHS is due to the fact that the condensate is a vacuum expectation value and so it is a sum over occupied negative energy states; and with m positive, the only negative energy threshold states correspond to species $\{2\}$, for which the Γ^0 sub-block is $-\gamma^0$. Changing the sign of m corresponds to interchanging the two species, so one has instead $+\gamma^0$. Thus, the condensate should be more precisely written as

$$\langle 0|\bar{\Psi}(\vec{x})\Psi(\vec{x})|0\rangle|_{m\rightarrow 0} = -\text{sign}(m) \sum_p |f_p^{(0)}(\vec{x})|^2 \quad (17)$$

Note that the condensate is determined entirely by the threshold states, which solve the first-order equation (13). We now consider several examples in which this condensate may be computed explicitly.

We begin with the familiar case of a uniform background magnetic field [5]. There is still gauge freedom of how we choose to represent the corresponding vector potential. This choice of gauge will determine the precise form of the threshold condition (13) (as well as the eigenvalue equation (10) which determines the complete spectrum). In the ‘linear gauge’, with $\vec{A} = (0, Bx)$, the threshold state equation (13) has normalized solutions

$$f_p^{(0)}(x, y) = \left(\frac{B}{\pi}\right)^{1/4} e^{ipy} e^{-(p-Bx)^2/(2B)} \quad (18)$$

where the degeneracy label p takes continuous values corresponding to a plane wave in the y direction. Thus, it is trivial to evaluate the condensate to be

$$\langle 0|\bar{\Psi}(\vec{x})\Psi(\vec{x})|0\rangle|_{m\rightarrow 0} = -\text{sign}(m) \sqrt{\frac{B}{\pi}} \int \frac{dp}{2\pi} e^{-(p-Bx)^2/B} = -\text{sign}(m) \frac{B}{2\pi} \quad (19)$$

In the ‘radial gauge’, with $\vec{A} = \frac{B}{2}(-y, x)$, the threshold state equation (13) has normalized solutions

$$f_p^{(0)}(x, y) = \sqrt{\frac{1}{\pi p!} \left(\frac{B}{2}\right)^{p+1}} z^p e^{-B|z|^2/4} \quad (20)$$

where we have defined the complex variable $z = x + iy$, and the degeneracy label p now takes integer values $p = 0, 1, 2, \dots$. Once again, it is trivial to evaluate the condensate to be

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle_{m \rightarrow 0} = -\text{sign}(m) \frac{B}{2\pi} e^{-B|z|^2/2} \sum_{p=0}^{\infty} \left(\frac{B}{2}\right)^p \frac{|z|^{2p}}{p!} = -\text{sign}(m) \frac{B}{2\pi} \quad (21)$$

The answer is, of course, the same in each gauge. Also note that in each of these cases, (19) and (21), the condensate is independent of \vec{x} , as is expected for a uniform B field. We now turn to some less trivial cases in which the magnetic field is spatially *inhomogeneous*.

3 Inhomogeneous Magnetic Fields: Two Examples

In this Section we consider two illustrative examples of specific inhomogeneous background magnetic fields. The first example is in the ‘radial gauge’, for which we choose the gauge field to be

$$\vec{A} = (-\partial_y \phi, \partial_x \phi) \quad (22)$$

where $\phi = \phi(r)$ is some function only of the radial coordinate r . Then the magnetic field is radial, $B(r) = \frac{1}{r} \frac{d}{dr} (r \frac{d}{dr} \phi)$, and the (un-normalized but mutually orthogonal) solutions to the threshold condition (13) are

$$f_p^{(0)} = z^p e^{-\phi} \quad (23)$$

where p is a non-negative integer.

We now choose a particular functional form for ϕ which will permit the explicit normalization of these states:

$$\phi(r) = \frac{BR^2}{4} \log \left(1 + \frac{r^2}{R^2} \right) \quad (24)$$

The corresponding radial magnetic field is

$$B(r) = \frac{B}{\left(1 + \frac{r^2}{R^2}\right)^2} \quad (25)$$

which has a *finite* net flux

$$\Phi \equiv \frac{1}{2\pi} \int d^2x B(r) = \frac{BR^2}{2} \quad (26)$$

The constant B represents the maximum value of the magnetic field, and R is a characteristic length scale associated with the spatial variation of the magnetic field. In the infinite flux limit, with $R \rightarrow \infty$, this example reduces to the constant B field example.

With this radial choice (24) for ϕ , the threshold states in (23) may be normalized, yielding

$$f_p^{(0)}(x, y) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{R^2} \right)^{(p+1)/2} \frac{1}{\sqrt{\beta(p+1, \Phi-p-1)}} \frac{z^p}{\left(1 + \frac{r^2}{R^2}\right)^{\Phi/2}} \quad (27)$$

where $\beta(u, v) \equiv \Gamma(u)\Gamma(v)/\Gamma(u+v)$ is the beta function. Having finite flux, this system displays the novel feature that only a finite number of these states are localized and normalizable. Indeed, these states are only localized for $p < [\Phi]$, where $[\Phi]$ is the largest integer less than Φ . The contribution to the condensate from these ‘bound’ states ² is

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} = -\text{sign}(m) \frac{1}{\pi R^2} \frac{1}{\left(1 + \frac{r^2}{R^2}\right)^\Phi} \sum_{p=0}^{[\Phi]} \frac{\left(\frac{r}{R}\right)^{2p}}{\beta(p+1, \Phi-p-1)} \quad (28)$$

When Φ is an integer we can in fact evaluate this sum exactly, yielding

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} = -\text{sign}(m) \left(1 - \frac{1}{\Phi}\right) \frac{1}{2\pi} \frac{B}{\left(1 + \frac{r^2}{R^2}\right)^2} \quad (29)$$

Note that the condensate has the same form as the magnetic field, but with an overall multiplicative factor depending on the net magnetic flux Φ . For large Φ this factor approaches unity, so that

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} \sim -\text{sign}(m) \frac{B(r)}{2\pi} \quad (30)$$

For our second example, consider a magnetic field that is uniform in one direction (say the y -direction), and spatially varying in the x direction. To achieve this type of configuration, we choose a convenient ‘linear gauge’ with $\vec{A} = (0, a(x))$. The corresponding magnetic field, $B(x) = a'(x)$, is just a function of x , and the (un-normalized but mutually orthogonal) solutions to the threshold equation (13) are

$$f_p^{(0)} = e^{ipy} e^{-\int^x (a(x)-p)} \quad (31)$$

The specific choice $a(x) = \lambda B \tanh(x/\lambda)$ yields a magnetic field profile

$$B(x) = \frac{B}{\cosh^2(x/\lambda)} \quad (32)$$

²The *integrated* condensate is proportional to the net magnetic flux Φ , with both localized and continuum states contributing [8, 9, 10, 11, 12]. Here, for a local analysis of the condensate *density* we only consider the localized bound states, in part because the continuum states contribute at infinity, and also because in the large flux limit the magnitude of their contribution is negligible.

The corresponding flux is

$$\Phi \equiv \frac{1}{2\pi} \int d^2x B = \frac{BL\lambda}{\pi} \quad (33)$$

where we have compactified the y -direction with a length L . For this type of magnetic field it is, in fact, possible to solve equation (10) for the *entire* spectrum, permitting for example the *exact* evaluation of the effective energy [15]. Here, however, for the evaluation of the vacuum condensate, we only need the normalized threshold states

$$f_p^{(0)}(x, y) = \sqrt{\frac{2}{L\lambda}} \frac{1}{\sqrt{\beta(B\lambda^2 + p\lambda, B\lambda^2 - p\lambda)}} e^{ipy} \frac{e^{px}}{(2\cosh(x/\lambda))^{B\lambda^2}} \quad (34)$$

With the y -direction compactified, the degeneracy label p takes discrete values $p = \frac{2\pi k}{L}$, where k is an integer such that

$$|k| < \frac{BL\lambda}{2\pi} \equiv \frac{\Phi}{2} \quad (35)$$

in order for these states to decay at infinity. We can therefore perform the sum in (17), yielding

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle_{m \rightarrow 0} = - \frac{2\text{sign}(m)}{L\lambda (2\cosh(x/\lambda))^{2B\lambda^2}} \sum_{k=-\lceil \frac{BL\lambda}{2\pi} \rceil}^{\lceil \frac{BL\lambda}{2\pi} \rceil} \frac{e^{4\pi k(x/L)}}{\beta(B\lambda^2 + 2\pi k \frac{\lambda}{L}, B\lambda^2 - 2\pi k \frac{\lambda}{L})} \quad (36)$$

As $L \rightarrow \infty$ we can replace the sum over k by an integral and find

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle_{m \rightarrow 0} = -\text{sign}(m) \frac{B}{\pi} \frac{1}{(2\cosh(x/\lambda))^{2B\lambda^2}} \int_{-1}^1 dt \frac{e^{2B\lambda^2 t(x/\lambda)}}{\beta(B\lambda^2(1+t), B\lambda^2(1-t))} \quad (37)$$

It is straightforward to plot this condensate for various values of the dimensionless combination $B\lambda^2$. One finds that the condensate has the same general ‘bell-shaped’ form as $-\text{sign}(m)B(x)/2\pi$, but that the correspondence is not exact. Nevertheless, for large $B\lambda^2$ (which corresponds to large flux) we can use Stirling’s formula to make an asymptotic expansion of the inverse beta function in the integrand of (37) to obtain

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle_{m \rightarrow 0} \sim -\text{sign}(m) \frac{B}{2\pi} \sqrt{\frac{B\lambda^2}{\pi}} \frac{1}{(\cosh(x/\lambda))^{2B\lambda^2}} \int_{-1}^1 dt \sqrt{1-t^2} \exp \left[B\lambda^2 \left(2t \left(\frac{x}{\lambda} \right) - \log(1-t^2) - t \log \left(\frac{1+t}{1-t} \right) \right) \right] \quad (38)$$

The remaining integral over t is suited for an asymptotic expansion, for large $B\lambda^2$, using Laplace’s method [16]. For an integral of the form

$$I(N) = \int ds \Psi(s) \exp(N\Omega(s)) \quad (39)$$

the large N leading asymptotic behavior is dominated by a critical value s_c at which the exponent function $\Omega(s)$ has a maximum, and is given by

$$I(N) \sim \sqrt{\frac{2\pi}{-N\Omega''(s_c)}} \Psi(s_c) \exp(N\Omega(s_c)) \quad (40)$$

Applying Laplace's method to the t integral in (39), for which the critical point is at $t_c = \tanh(x/\lambda)$, we find the condensate to be asymptotically proportional to the inhomogeneous magnetic field:

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} \sim -\text{sign}(m) \frac{B}{2\pi} \frac{1}{(\cosh(x/\lambda))^2} = -\text{sign}(m) \frac{B(x)}{2\pi} \quad (41)$$

4 Asymptotic Analysis for General $B(r)$

At first sight, one might think that the asymptotic proportionality between the condensate and the inhomogeneous magnetic field found in (30) and (41) is due to the special form of the inhomogeneous magnetic field chosen in these examples. For example, in each of these cases the normalization factors may be computed exactly and are given by beta functions. In general it is not possible to compute the normalization factors in closed form. However, we show in this Section that this asymptotic analysis applies to very general inhomogeneous magnetic fields $B(r)$ and $B(x)$. As the analysis is very similar in the two cases, we concentrate on the radial case. In the conclusion we make some comments concerning the general $B(x, y)$ case.

We choose the gauge field in the radial gauge (22) with $\phi = \phi(r)$. It is convenient to write

$$\phi(r) = \frac{BR^2}{4} h\left(\frac{r^2}{R^2}\right) \quad (42)$$

where R is some characteristic length scale, and B together with the dimensionless function h are chosen so that the overall normalization gives net flux $\Phi = \frac{BR^2}{2}$. The special cases $h(\xi) = \xi$ and $h(\xi) = \log(1 + \xi)$ have been considered in the preceeding Sections. Then, as before, the threshold states are given by (23). Note that these are automatically mutually orthogonal for any $\phi(r)$ by virtue of the angular integration. The vacuum condensate is

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} = -\text{sign}(m) \sum_{p=0}^{[\Phi]} \frac{(r/R)^{2p} e^{-\Phi h(r^2/R^2)}}{N_p^2} \quad (43)$$

where the normalization factors N_p^2 are given by

$$N_p^2 = \pi R^2 \int_0^\infty d\left(\frac{r^2}{R^2}\right) \left(\frac{r}{R}\right)^{2p} e^{-\Phi h(r^2/R^2)} \quad (44)$$

For large flux, the sum over p in (43) may be replaced by an integral from 0 to Φ , which may be re-expressed in terms of the rescaled variable $t = p/\Phi$ as

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} = -\text{sign}(m) \Phi \int_0^1 dt \left(\frac{e^{\Phi(t \log \xi - h(\xi))}}{N_t^2} \right) \quad (45)$$

with normalization factors

$$N_t^2 = \pi R^2 \int_0^\infty d\xi e^{\Phi(t \log \xi - h(\xi))} \quad (46)$$

where $\xi = \frac{r^2}{R^2}$.

To evaluate the asymptotic form of the condensate (45) for large flux Φ we first need the asymptotic form of the normalization factors for large Φ . This can also be done by Laplace's method, with the dominant contribution coming from the maximum of the exponent function

$$\chi(\xi) = t \log \xi - h(\xi) \quad (47)$$

This defines a critical point $\xi_c = \xi_c(t)$, as a function of the parameter t , via the implicit relation

$$\xi h'(\xi) = t \quad (48)$$

Using (40) we see that

$$N_t^2 \sim \pi R^2 \sqrt{\frac{2\pi}{\Phi}} e^{\Phi[t \log \xi_c(t) - h(\xi_c(t))]} \frac{1}{\sqrt{\frac{t}{\xi_c(t)^2} + h''(\xi_c(t))}} \quad (49)$$

Inserting this into the integrand in (45) we obtain

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} \sim -\text{sign}(m) \frac{B}{2\pi} \sqrt{\frac{\Phi}{2\pi}} \int_0^1 dt \sqrt{\frac{t}{\xi_c(t)^2} + h''(\xi_c(t))} e^{\Phi[t \log(\xi/\xi_c(t)) - h(\xi) + h(\xi_c(t))]} \quad (50)$$

The remaining t integral may also be expanded asymptotically using Laplace's method, with exponent function

$$\Omega(t) = t \log \xi - t \log \xi_c(t) - h(\xi) + h(\xi_c(t)) \quad (51)$$

This leads to a maximum at the critical point $t_c = t_c(\xi)$ as a function of $\xi = r^2/R^2$, defined by the relation $\xi = \xi_c(t_c)$. Applying the inverse function theorem (see comments below equation (54)), we find

$$t_c = \xi h'(\xi) \quad (52)$$

and

$$-\Omega''(t_c) = \frac{1}{\xi(h'(\xi) + \xi h''(\xi))} \quad (53)$$

Combining all these pieces, several remarkable cancellations occur, and the leading asymptotic form of (50) is simply given by

$$\begin{aligned} \langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} &\sim -\text{sign}(m) \frac{B}{2\pi} (h'(\xi) + \xi h''(\xi)) \\ &= -\text{sign}(m) \frac{B(r)}{2\pi} \end{aligned} \quad (54)$$

This result depends on subtle cancellations between the asymptotic expansion of the (inverse of the) normalization integral (which is an integral over r or equivalently ξ) and the asymptotic expansion of the sum over the threshold states (which becomes an integral over t). These cancellations rely on the use of the inverse function theorem which assumes that the function $\xi h'(\xi)$ is one-to-one (see (48)). This means that its derivative, $(\xi h'(\xi))'$, has a fixed sign, which moreover must be positive in order for the critical point to be a maximum. Since $(\xi h'(\xi))'$ is just the magnetic field, we see that our result applies over a region of space in which the inhomogeneous magnetic field is positive. This is the relevant physical set-up and is consistent with our intuitive expectation - in a region with positive magnetic field, the spin density tends to align with the magnetic field; moreover, if the flux in this region is very large, the *local* spin density will actually be approximately proportional to the *local* inhomogeneous magnetic field. The proportionality factor is such that when this result is integrated over the region, we regain the Aharonov-Casher relation [6, 7] between the integrated spin and the net magnetic flux.

5 Concluding Remarks

In this paper we have studied the vacuum condensate of parity invariant $2 + 1$ dimensional QED in the presence of inhomogeneous magnetic fields. After presenting two explicit illustrative examples in which the condensate may be evaluated in detail, we showed that in the limit of large flux the condensate is *locally* proportional to the inhomogeneous magnetic field:

$$\langle 0 | \bar{\Psi}(\vec{x}) \Psi(\vec{x}) | 0 \rangle |_{m \rightarrow 0} \sim \begin{cases} -\text{sign}(m) \frac{B(r)}{2\pi} \\ -\text{sign}(m) \frac{B(x)}{2\pi} \end{cases} \quad (55)$$

for general physically relevant magnetic fields which either depend only on the radial coordinate or only on one of the Cartesian coordinates. These relations represent a *local* analogue of the *integrated* Aharonov-Casher relation. To generalize this result to general static magnetic fields $B(x, y)$, we note that the background vector potential may be represented as

$$\vec{A} = (-\partial_y \phi(x, y), \partial_x \phi(x, y)) \quad (56)$$

where

$$\nabla^2 \phi(x, y) = B(x, y) \quad (57)$$

Then the threshold solutions are simply

$$f_p(x, y) = F_p(z) e^{-\phi(x, y)} \quad (58)$$

where $F_p(z)$ is some holomorphic function. However, to proceed one must construct an *orthogonal basis* of such holomorphic functions. When $\phi = \phi(r)$ this may be achieved by the choice $F_p(z) = z^p$, as in (23), and when $\phi = \phi(x)$ by the choice $F_p(z) = e^{pz}$, as in (31). For a general $\phi = \phi(x, y)$ the natural choice of orthogonal basis is not so clear (a Gramm-Schmidt orthogonalization is too clumsy for the subsequent summation). Further clarification of this issue should lead to interesting insights into the properties of vacuum condensates of $2 + 1$ dimensional QED with dynamical gauge fields.

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